Appendix A

Big-Oh and Little-Oh

The notation *Big-Oh* and *Little-Oh* is used to compare the relative values of two functions, $f(x)$ and $g(x)$, as x approaches ∞, or 0, depending on which of these two cases is being considered. We will suppose that g is positive-valued and that $x > 0$.

A.1 Big-Oh

The case $\mathbf{x \to \infty}$

f is $\mathcal{O}(g)$ if there exist constants $A > 0$ and $x_0 > 0$ such that $\dfrac{|f(x)|}{g(x)} < A$ for all $x > x_0$.

The case $\mathbf{x \to 0}$

f is $\mathcal{O}(g)$ if there exist constants $A > 0$ and $x_0 > 0$ such that $\dfrac{|f(x)|}{g(x)} < A$ for all $x < x_0$.

Example A1

The function $f(x) = 3x^3 + 4x^2$ is $\mathcal{O}(x^3)$ as $x \to \infty$. (Here $g(x) = x^3$.) We have that

$$\frac{|f(x)|}{g(x)} = \frac{3x^3 + 4x^2}{x^3} = 3 + \frac{4}{x} \, .$$

There exist (infinitely) many pairs $A, x_0 > 0$ that show that f is $\mathcal{O}(x^3)$, for example, $\frac{|f(x)|}{g(x)} < 4.1$ for all $x > 40$.

Example A2

The function $f(x) = 3x^3 + 4x^2$ is $\mathcal{O}(x^2)$ as $x \to 0$. Here

$$\frac{|f(x)|}{g(x)} = \frac{3x^3 + 4x^2}{x^2} = 3x + 4 \, ,$$

and, for example, $\frac{|f(x)|}{g(x)} < 4.3$ for all $x < 0.1$.

A.2 Little-Oh

The case $\mathbf{x} \to \infty$: f is $o(g)$ if $\lim_{x\to\infty} \frac{|f(x)|}{g(x)} = 0\,.$

The case $\mathbf{x} \to \mathbf{0}$: f is $o(g)$ if $\lim_{x\to 0} \frac{|f(x)|}{g(x)} = 0\,.$

Example A3

The function $f(x) = 3x^3 + 4x^2$ is $o(x^4)$ as $x \to \infty$ because

$$\lim_{x\to\infty} \frac{|f(x)|}{g(x)} = \lim_{x\to\infty} \frac{3x^3 + 4x^2}{x^4} = \lim_{x\to\infty} \left(\frac{3}{x^2} + \frac{4}{x} \right) = 0\,.$$

Example A4

The function $f(x) = 3x^3 + 4x^2$ is $o(x)$ as $x \to 0$ because

$$\lim_{x\to 0} \frac{|f(x)|}{g(x)} = \lim_{x\to 0} \frac{3x^3 + 4x^2}{x} = \lim_{x\to 0} (3x^2 + 4x) = 0\,.$$

Example A5

The function $f(x) = 3x^2 + 4x$ is $o(1)$ as $x \to 0$ because

$$\lim_{x\to 0} \frac{|f(x)|}{g(x)} = \lim_{x\to 0} \frac{3x^2 + 4x}{1} = \lim_{x\to 0} (3x^2 + 4x) = 0\,.$$

Appendix B

Taylor expansions

This appendix gives a brief justification of the expansions used in Section 1.3.2. Details can be found in standard calculus texts, for example, Courant R. and John F. (1965) *Introduction to Calculus and Analysis*, Wiley, New York.

Suppose that the function f has $n+1$ continuous derivatives in the interval $[a, a+h]$, if $h > 0$, or $[a+h, a]$, if $h < 0$. The n-term Taylor approximation for $f(a+h)$ is given by

$$f(a+h) = f(a) + \frac{h}{1!}f^{(1)}(a) + \frac{h^2}{2!}f^{(2)}(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + R_n(a,h), \tag{B.1}$$

where $f^{(r)}(a)$ denotes the r-th derivative of f at the point a, and $R_n(a,h)$, called the remainder, is the error in approximating $f(a+h)$ by the n-degree polynomial (in h) on the right-hand side.

The remainder can be represented in different forms. The following result is based on a version of the *Lagrange form*[1]: If there exists a positive constant M, such that $|f^{(n+1)}(t)| \leq M$ for all $t \in [a, a+h]$ (or alternatively $t \in [a+h, a]$, in the case $h < 0$) then

$$|R_n(a,h)| \leq \frac{|h|^{n+1}}{(n+1)!}\, M\,.$$

Thus, regarded as a function of n, for fixed a and h, the term $|R_n(a,h)|$ becomes small as n increases. Alternatively, regarded as a function of h, for fixed a and n, $|R_n(a,h)|$ becomes small as $h \to 0$. In the notation explained in Appendix A, $R_n(a,h)$ is $o(h^n)$ as $h \to 0$.

Example B1

This example relates to the material in Section 1.3.2. Specifically, we wish to investigate the behaviour of a two-term Taylor approximation to $f(x - zh)$ as h becomes small, for fixed values x and z. We assume that f is three times differentiable, and that, in some

[1] After the Italian-French mathematician Joseph-Louis Lagrange (1736-1813).

closed interval containing x, the absolute value of it's third derivative is bounded by some positive constant M. Applying the expansion (B.1) yields

$$f(x - hz) = f(x) + \frac{(-hz)}{1!} f'(x) + \frac{(-hz)^2}{2!} f''(x) + R_2(x, -zh),$$

where $|R_2(x, -zh)| \leq \frac{|zh|^3}{3!} M$. Thus as, h becomes small, we have that

$$f(x - hz) = f(x) - hz f'(x) + \frac{h^2 z^2}{2} f''(x) + o(h^2)$$

Similarly it follows that

$$f(x - hz) = f(x) - hz f'(x) + o(h)$$

and that

$$f(x - hz) = f(x) + o(1)$$